1021. Proposed by George Apostolopoulos, Messolonghi, Greece.

Let ABC be an acute triangle. Show that

$$\sum \frac{\sec A}{\sqrt{\cos A + \cos B}} \geq 6,$$

where the sum $\sum$ is over all cyclic permutations of .(A, B, C).

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Note that by AM-GM Inequality ($\cos A, \cos B > 0$) we have

$\sqrt{\cos A + \cos B} = \sqrt{1(\cos A + \cos B)} \leq \dfrac{1 + \cos A + \cos B}{2}$

Then $\displaystyle\sum \frac{\sec A}{\sqrt{\cos A + \cos B}} \geq \sum \frac{2\sec A}{1 + \cos A + \cos B} = \sum \frac{2}{\cos A + \cos^2 A + \cos A \cos B}$.

Since by Cauchy Inequality

$\displaystyle\sum \frac{1}{\cos A + \cos^2 A + \cos A \cos B} \geq \frac{9}{\sum(\cos A + \cos^2 A + \cos A \cos B)}$ then suffice to prove

$\displaystyle\frac{18}{\sum(\cos A + \cos^2 A + \cos A \cos B)} \geq 6 \Leftrightarrow 3 \geq \sum \cos A + \sum \cos^2 A + \sum \cos A \cos B \Leftrightarrow$

$\sum(1 - \cos^2 A) \geq \sum \cos A + \sum \cos A \cos B \Leftrightarrow$

(**1**) $\qquad \sum \sin^2 A \geq \sum \cos A + \sum \cos A \cos B.$

Noting that $\sum \cos A = 1 + \dfrac{r}{R}$, $\sum \cos A \cos B = \dfrac{s^2 + r^2 - 4R^2}{4R^2}$, $\sum \sin^2 A = \dfrac{a^2 + b^2 + c^2}{4R^2} =$

$\dfrac{2(s^2 - 4Rr - r^2)}{4R^2}$ we can rewrite inequality (**1**) as

$\dfrac{2(s^2 - 4Rr - r^2)}{4R^2} \geq 1 + \dfrac{r}{R} + \dfrac{s^2 + r^2 - 4R^2}{4R^2} \Leftrightarrow 2(s^2 - 4Rr - r^2) \geq 4Rr + s^2 + r^2 \Leftrightarrow$

(**2**) $\quad s^2 \geq 12Rr + 3r^2.$

Inequality (2) follows immediately from well known Gerretsen's Inequality $s^2 \geq 16Rr - 5r^2$ and Euler's Inequality $R \geq 2r$. Indeed,

$s^2 - (12Rr + 3r^2) = (s^2 - 16Rr + 5r^2) + ((16Rr - 5r^2) - (12Rr + 3r^2)) =$

$(s^2 - 16Rr + 5r^2) + 4r(R - 2r) \geq 0.$

Inequality (2) can be easy proved directly, without reference to Gerretsen's Inequality an Euler's Inequality, if we use "free parametrization of a triangle", namely let $x = s - a, y = s - b, z = s - c$. Then assuming $s = 1$, due to homogeneity of inequality (2), we obtain $a = 1 - x, b = 1 - y, c = 1 - z$, where $x + y + z = 1, x, y, z > 0$, and since

$4Rr = 4Rr = abc = (1 - x)(1 - y)(1 - z) = xy + yz + zx - xyz, r^2 = \dfrac{(s - a)(s - b)(s - c)}{s} =$

xyz we obtain that (**1**)$\Leftrightarrow 1 \geq 3(xy + yz + zx - xyz) + 3xyz \Leftrightarrow$

$3(xy + yz + zx) \leq 1 = (x + y + z)^2.$